「名古屋大学HPC計算科学連携研究プロジェクト」
シンポジウム　5/10/2011

地球流体乱流の数値解析

木村　芳文

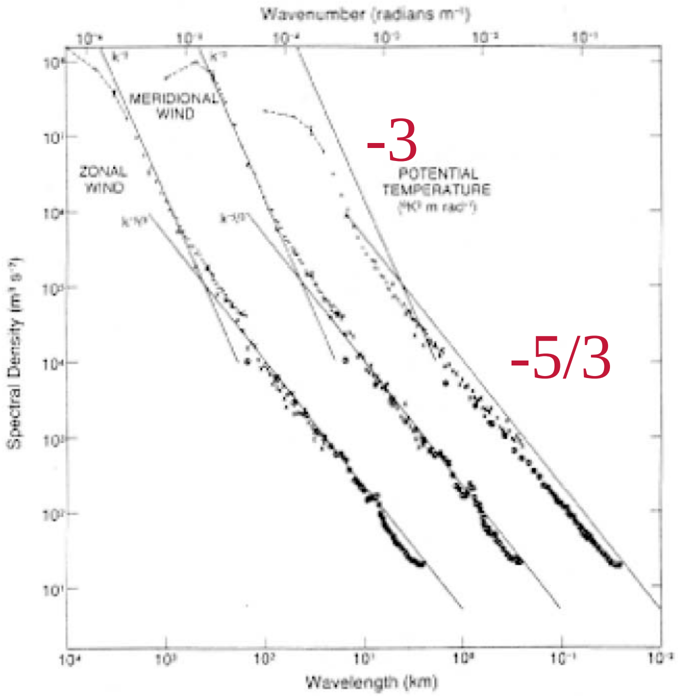
名古屋大学多元数理科学研究科

collaboration:

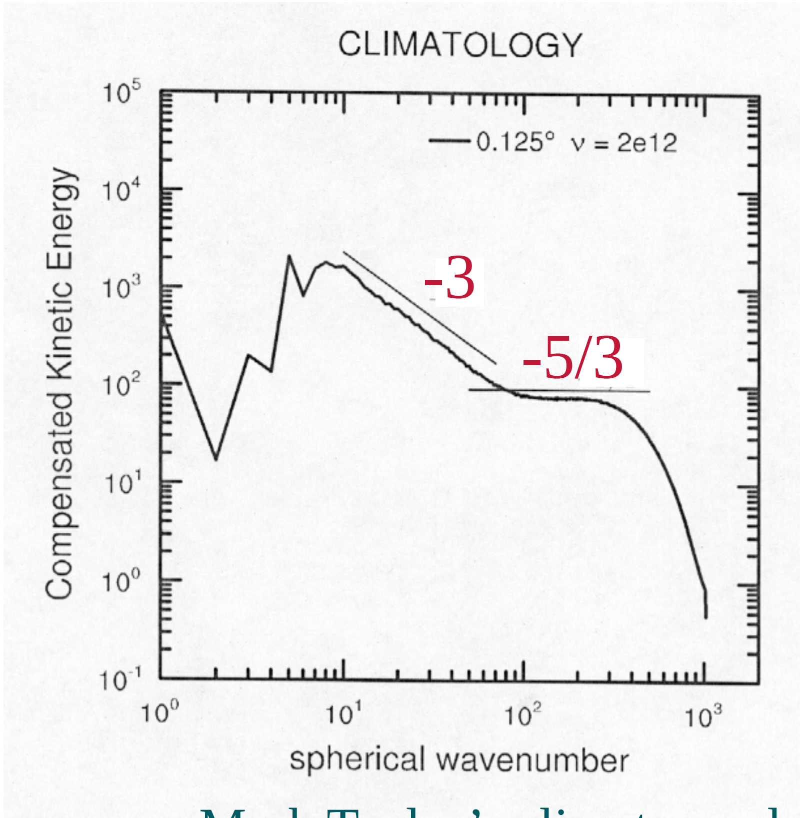
Jackson R. Herring

National Center for Atmospheric Research

Transition in Energy Spectrum for Rotating and Stratified turbulence



Nastrom-Gage's atmospheric observation (1985) (*JAS* **42** 950-960.)



Mark Taylor's climate model simulation (2008)
(CCSM project at NCAR)

Both stratification and rotation are essential

-3 : enstrophy cascade for Quasi-Geostrophic turbulence (~2D)
-5/3 : Kolmogorov turbulence (3D)

Transition in Energy Spectrum for Stratified turbulence

Observations: (in the ocean)

$k_z^{-2 \sim -3}$ → $k^{-5/3}$

Garret-Munk spectrum → Kolmogorov spectrum

Munk (1981), Garrett *et.al* (1981)

transition wavenumbe: $k_c \sim \sqrt{N^3/\varepsilon}$ (Ozmidov scale)

Theory: Munk (1981), Garrett *et.al* (1981), Lumley (1964), Holloway (1983)

All support the Ozmidov scale for transition

Simulation:

Carnevale, Briscoline & Orlandi (2001) LES at 128^3

Yoshida, Ishihara & Kaneda (2002) LES up to 512^3

~ Ozmidov for transition

Waite & Bartello (2004) DNS + hyperviscosity

(Waite & Bartello (2004) for the review)

Navier-Stokes equation with the Boussinesq approximation

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \nabla^2 \mathbf{u} + \theta \hat{\mathbf{z}} + \mathbf{F}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{u} \cdot \nabla)\theta = \kappa \nabla^2 \theta - N^2 w$$

$$\nabla \cdot \mathbf{u} = 0$$

where

$\mathbf{u} = (u,\ v,\ w)$: velocity

θ : temperature fluctuations

$N^2 = \dfrac{g\alpha}{T_0} \dfrac{\partial \overline{T}}{\partial z}$: Brunt - Väisälä frequency

$\mathbf{F}$: Forcing (horizontal)

Numerical Methods

- forced simulations
- 2π-periodic box with 1024^3 grid points ($R_\lambda \sim 300$)
- 3^{rd} order time-marching scheme
- Initial energy spectrum : $E(k) = 0$
- Force horizontal velocity components
- Add red noise to modes within a wave number band ($k_f \sim 5$)

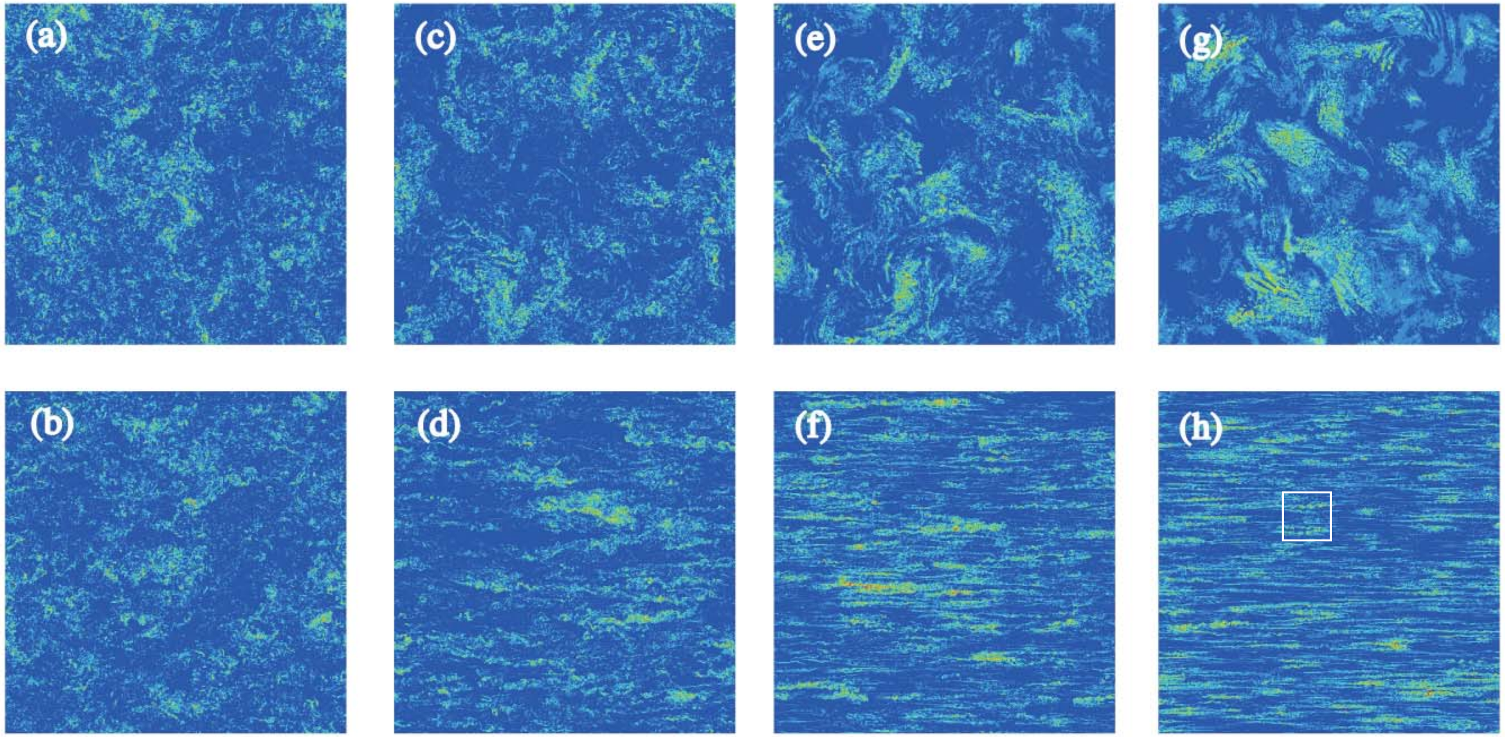
Solving Ornstein-Uhlenbeck process (2nd order stochastic ODEs)

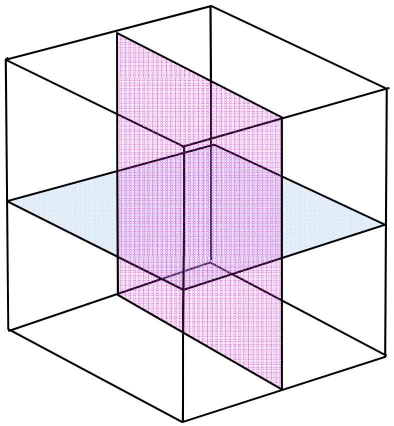
Enstrophy contours

$N^2=1$ $N^2=10$ $N^2=50$ $N^2=100$

horizontal

vertical

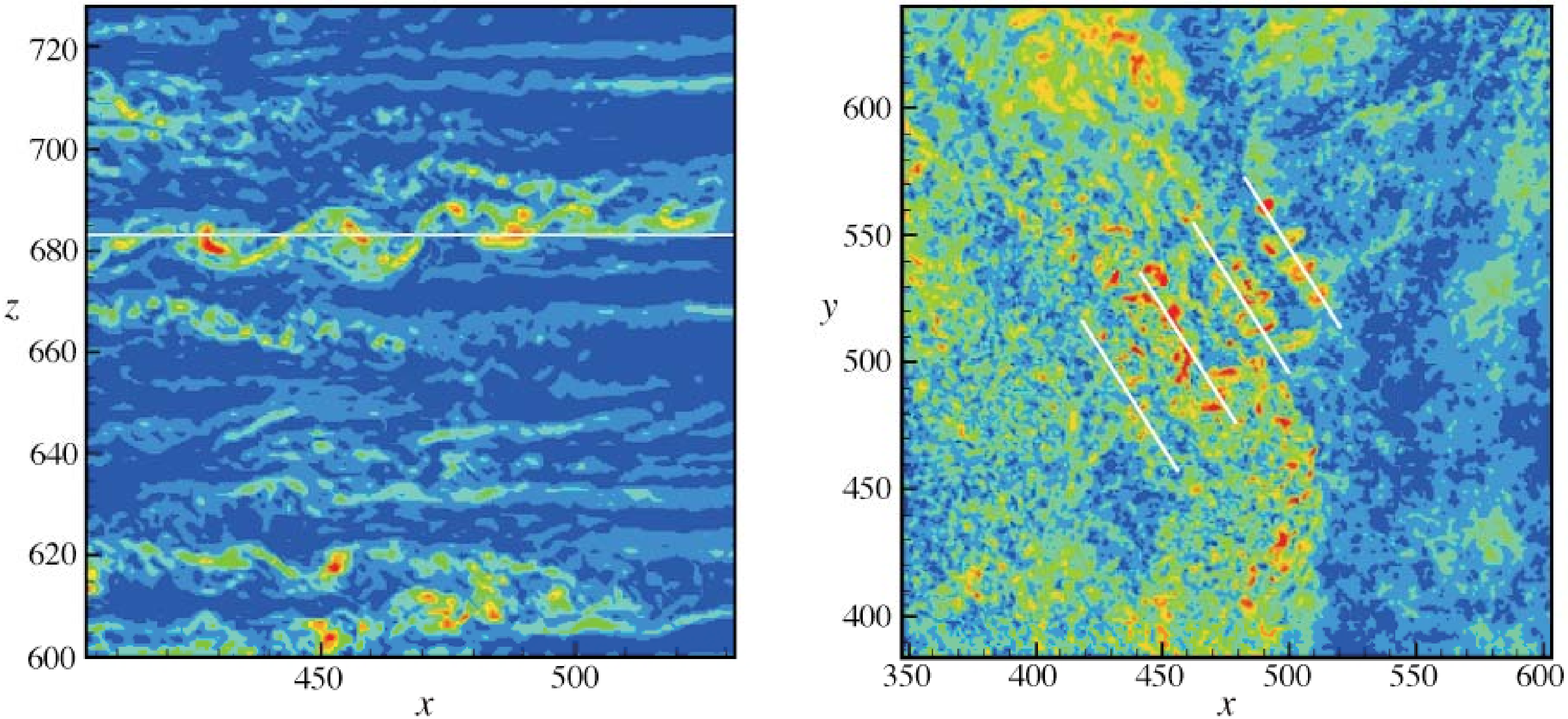




As N^2 becomes large;

- large scale clusters and elongated streaks appear in the horizontal plane.
- thin layers and wedge structures develop in the vertical plane.

Enstrophy contours (blow-up)



- Kelvin-Helmholz billows are observed in the vertical.
- The billows are not single rollers and chopped in the horizontal.

Characteristics of stratified turbulence

- Composite of “waves” and “turbulence”

  → “*Craya-Herring decomposition*” to separate waves and turbulence

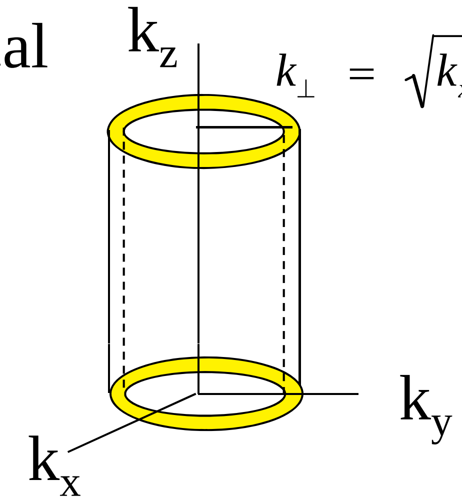
- Highly anisotropic

  → Need suitable avaraging

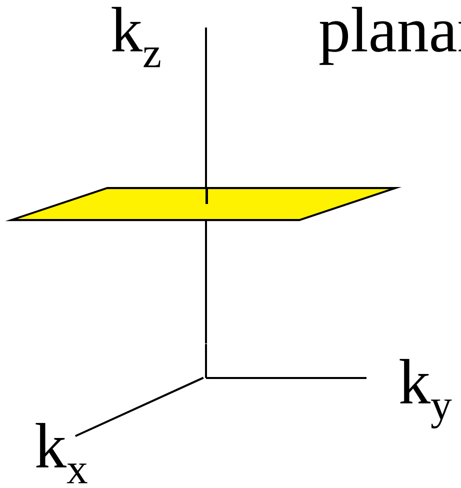
$$\mathbf{k} = k_\perp \otimes k_z$$

cylindrical

$$k_\perp = \sqrt{k_x^2 + k_y^2}$$



planar



"Craya-Herring" decomposition

$$\nabla\cdot\mathbf{u} = 0 \quad \Longleftrightarrow \quad \mathbf{k}\cdot\tilde{\mathbf{u}} = 0$$

incompressibility

$\tilde{\mathbf{u}}$ is spanned by two independent vectors perpendicular to **k**

$$\mathbf{e}_1(\mathbf{k}) = \frac{\mathbf{k}\times\hat{\mathbf{z}}}{\|\mathbf{k}\times\hat{\mathbf{z}}\|} = \frac{1}{\sqrt{k_x^2+k_y^2}}\begin{pmatrix} k_y \\ -k_x \\ 0 \end{pmatrix}$$

$$\mathbf{e}_2(\mathbf{k}) = \frac{\mathbf{k}\times\mathbf{k}\times\hat{\mathbf{z}}}{\|\mathbf{k}\times\mathbf{k}\times\hat{\mathbf{z}}\|} = \frac{1}{\sqrt{k_x^2+k_y^2+k_z^2}\sqrt{k_x^2+k_y^2}}\begin{pmatrix} k_z k_x \\ k_z k_y \\ -(k_x^2+k_y^2) \end{pmatrix}$$

$$\mathbf{e}_3(\mathbf{k}) = \frac{\mathbf{k}}{\|\mathbf{k}\|} = \frac{1}{\sqrt{k_x^2+k_y^2+k_z^2}}\begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix}$$

orthnormal coordinates

$$\tilde{\mathbf{u}}(\mathbf{k}) = \phi_1\mathbf{e}_1(\mathbf{k}) + \phi_2\mathbf{e}_2(\mathbf{k})$$

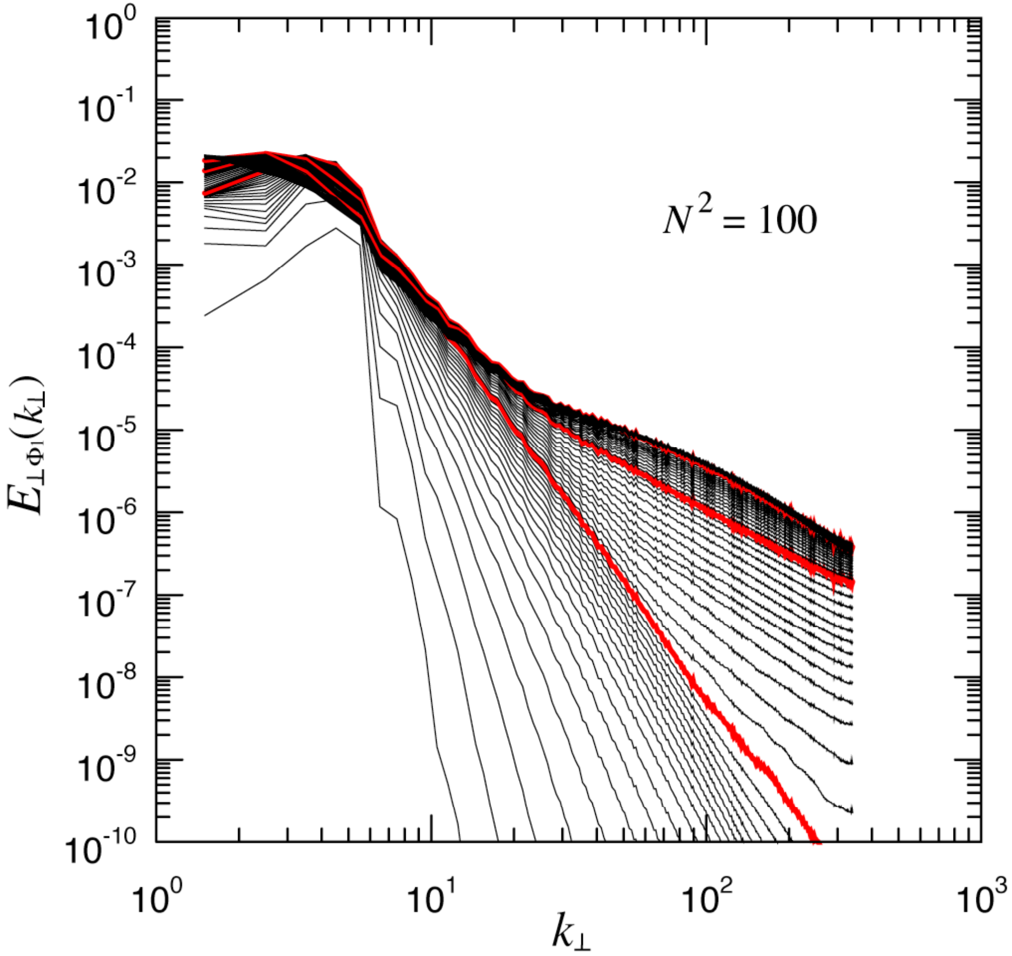
$$\phi_1 = \tilde{\mathbf{u}}(\mathbf{k})\cdot\mathbf{e}_1(\mathbf{k}) = \frac{1}{\sqrt{k_x^2+k_y^2}}(k_y\tilde{u}-k_x\tilde{v}) = \frac{i}{\sqrt{k_x^2+k_y^2}}\tilde{\omega}$$

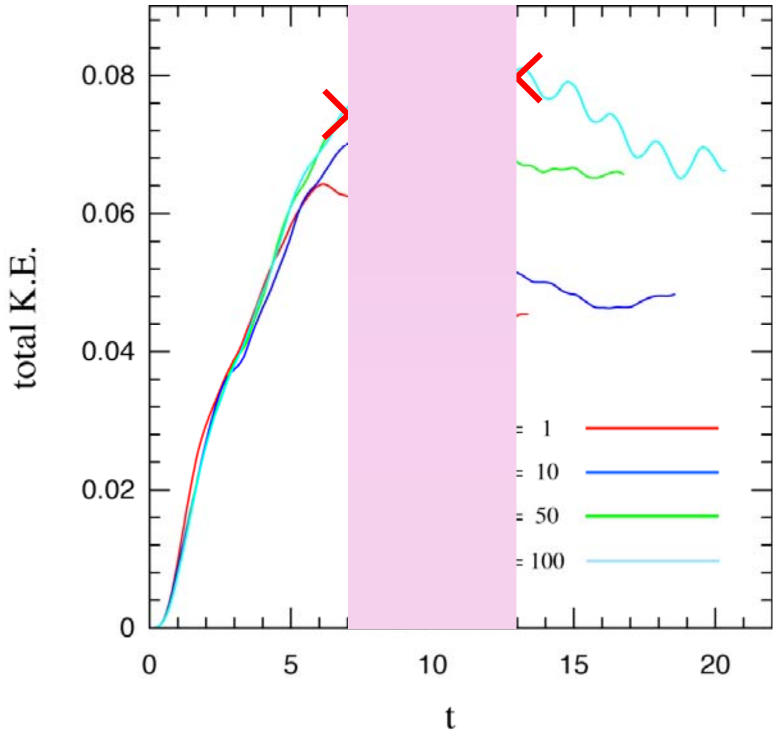
(vortex, rotation)

$$\phi_2 = \tilde{\mathbf{u}}(\mathbf{k})\cdot\mathbf{e}_2(\mathbf{k}) = \frac{\sqrt{k_x^2+k_y^2+k_z^2}}{\sqrt{k_x^2+k_y^2}}\tilde{w}$$

(wave, divergence)

History of Φ_1 energy spectra (N^2=100)





First, steep spectrum (~k^{-3}) develops then small scales rise.

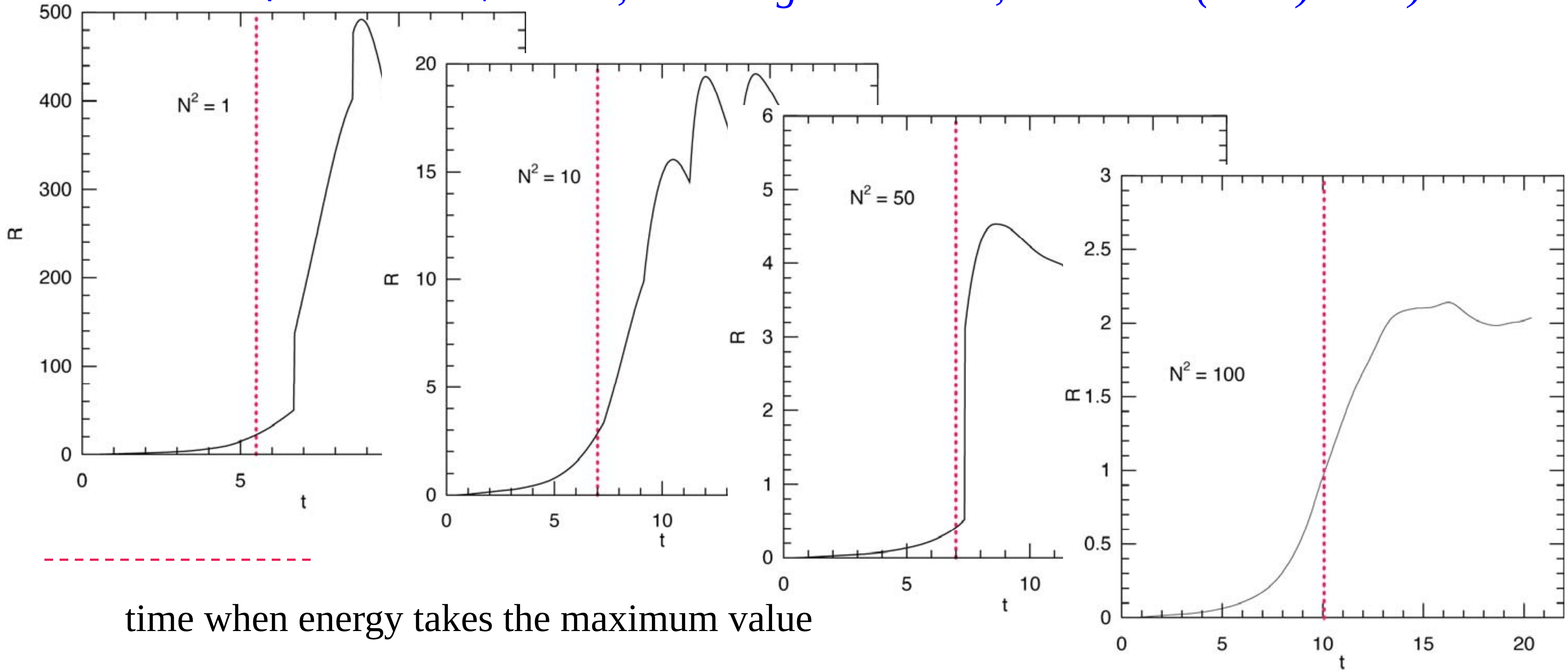
History of buoyancy Reynolds number

$$R = \mathrm{Fr}_h^2\,\mathrm{Re} = \frac{\varepsilon}{\nu N^2} \;\rightarrow\; \left[\sqrt{\frac{\varepsilon}{\nu N^2}} \Big/ \left(\nu^3/\varepsilon\right)^{1/4}\right]^{4/3} = \left[L_O/L_K\right]^{4/3}$$

L_O : Ozmidov scale L_K : Kolmogorov scale

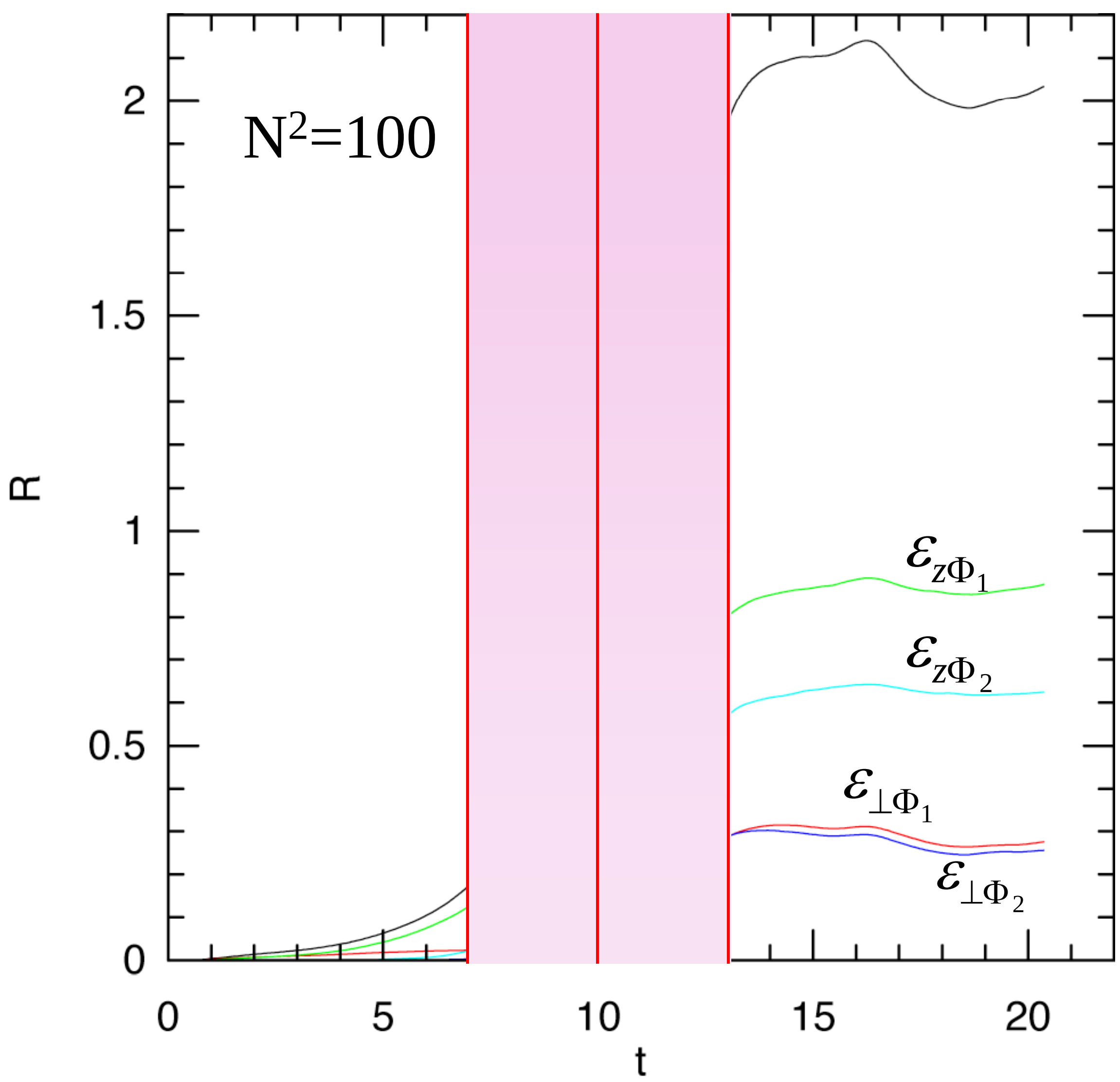
R < 1 : steep spectrum,   R > 1: -5/3.

(Brethouwer, Billant, Lindborg & Chomaz, JFM ***585*** *(2007) 343.)*

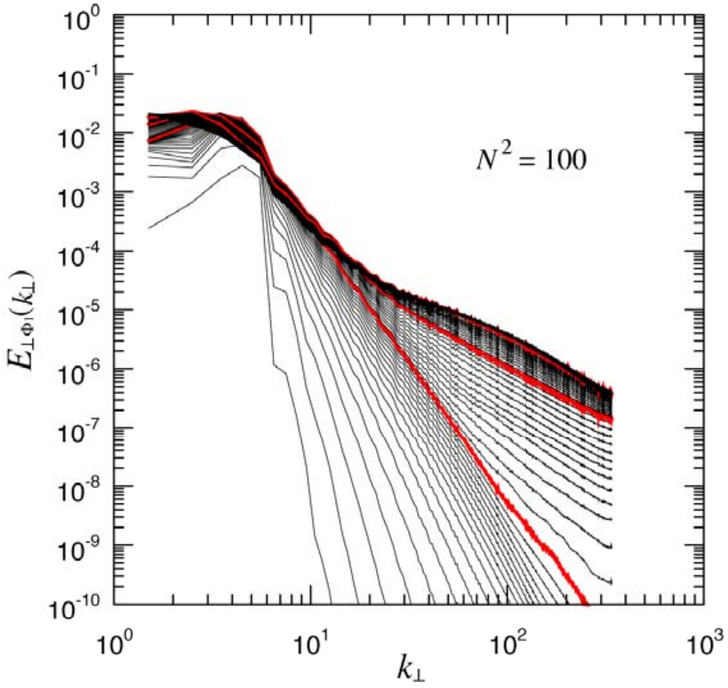


time when energy takes the maximum value

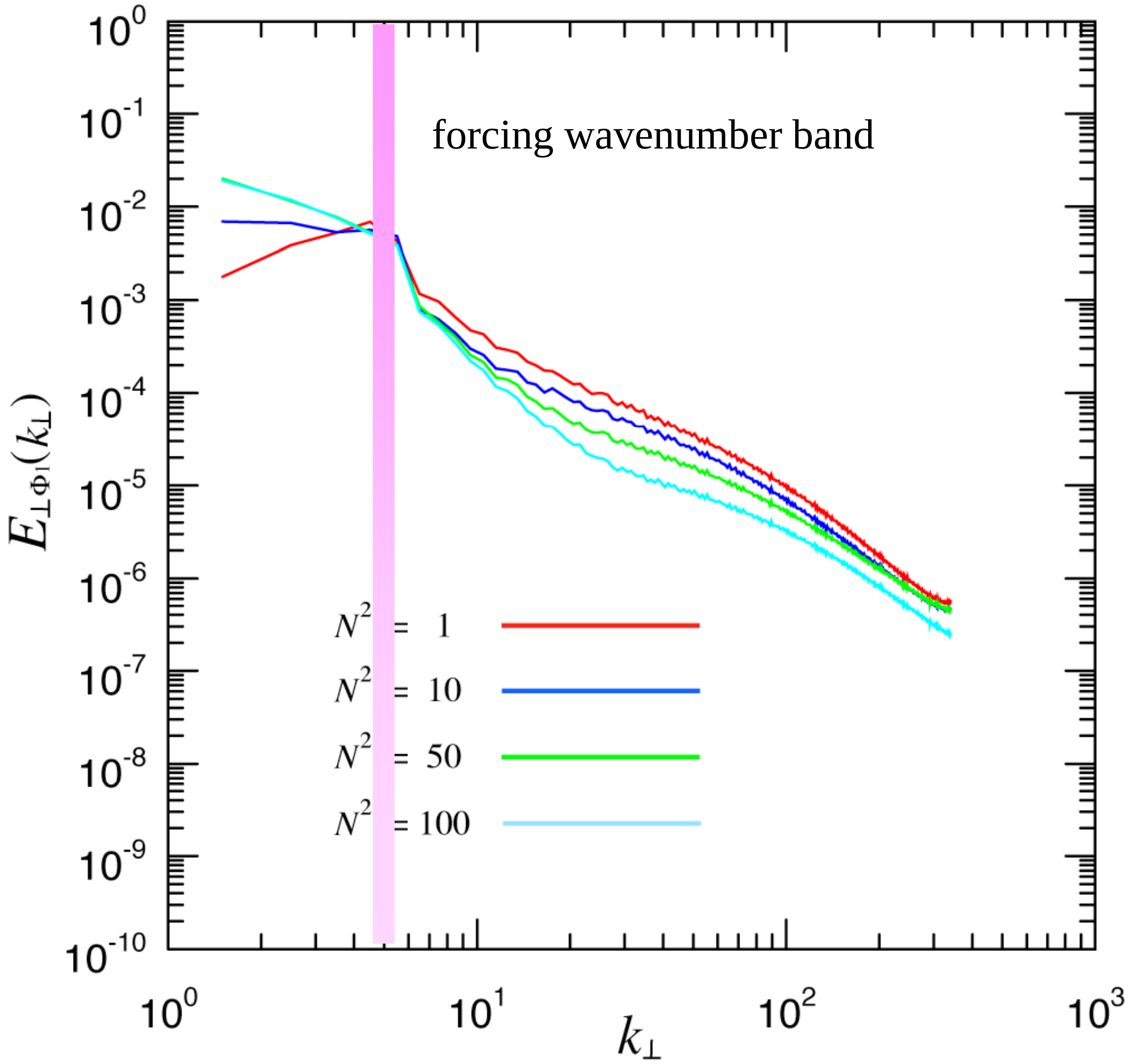
History of buoyancy Reynolds number



$$R = \mathrm{Fr}_h^2\,\mathrm{Re} = \frac{\varepsilon}{\nu N^2}$$



$\Phi_1(k_\perp)$ spcetra for various N



- -5/3 for small scales (different coefficient)
- -3 for large scales (same coefficient)
- There's a sharp transition

How to scale them ?

guidelines:

- The Kolmogorov const. is universal. (Sreenivasan 85)
- The coefficient for large scales doesn't depend on N

Anisotropy in dissipation

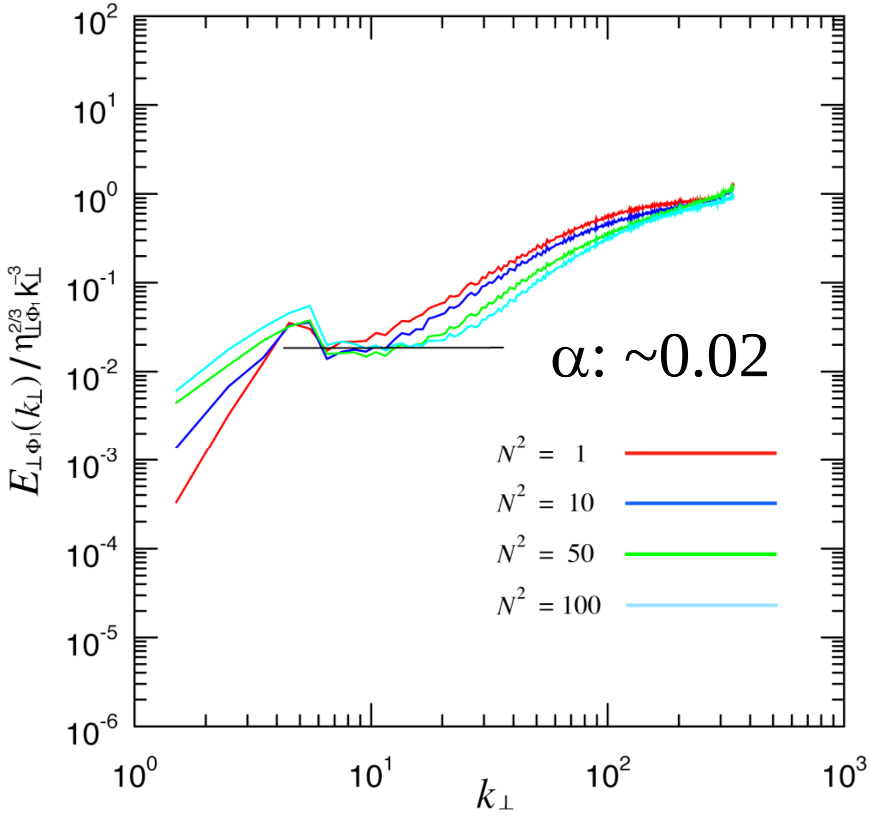
$$\varepsilon = 2\nu \int_0^\infty \mathbf{k}^2 E(\mathbf{k}) d\mathbf{k} = 2\nu \int_0^\infty (k_\perp{}^2 + k_z^2)\ (E_{\Phi_1} + E_{\Phi_2}) d\mathbf{k}$$

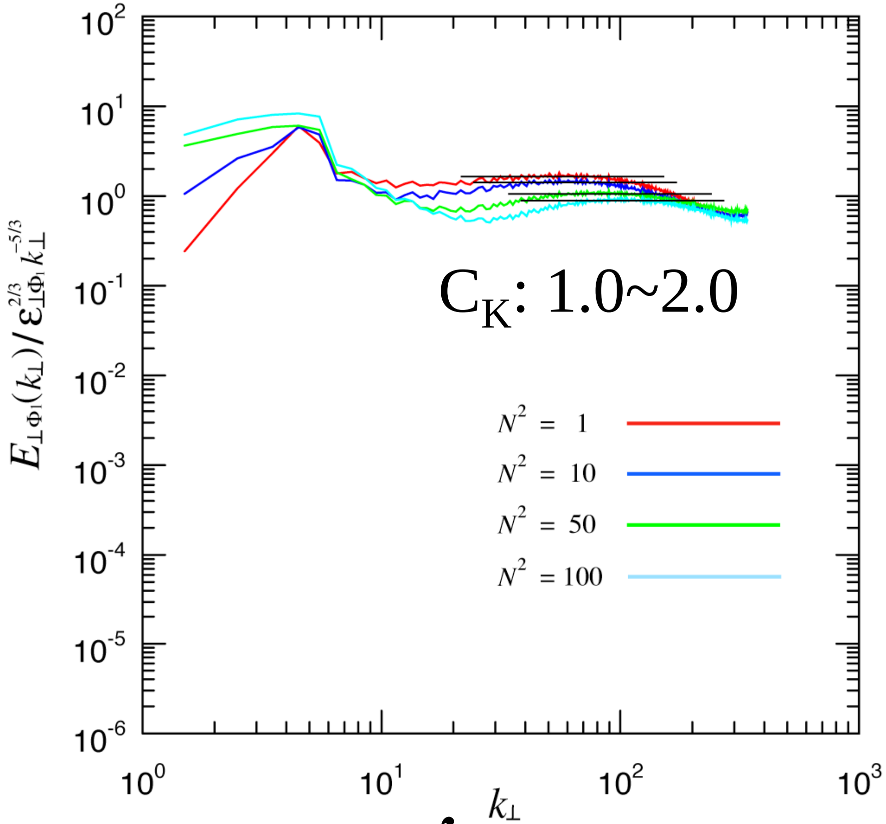
$$= \underbrace{\varepsilon_{\perp\Phi_1} + \varepsilon_{\perp\Phi_2}}_{\text{horizontal dissipation}} + \underbrace{\varepsilon_{z\Phi_1} + \varepsilon_{z\Phi_2}}_{\text{vertical dissipation}}$$

$\langle |\partial u / \partial z|^2 \rangle \longrightarrow \dfrac{N^2}{\langle |\partial u / \partial z|^2 \rangle}$ (local Richardson number)

N^2	$\varepsilon_{\perp\Phi_1}$	$\varepsilon_{\perp\Phi_2}$	$\varepsilon_{z\Phi_1}$	$\varepsilon_{z\Phi_2}$	ε
1	1.63×10^{-3}	1.72×10^{-3}	9.28×10^{-4}	9.45×10^{-4}	5.23×10^{-3}
10	1.24×10^{-3}	1.46×10^{-3}	1.12×10^{-3}	1.18×10^{-3}	5.01×10^{-3}
50	1.09×10^{-3}	1.10×10^{-3}	2.07×10^{-3}	1.74×10^{-3}	6.00×10^{-3}
100	6.61×10^{-4}	5.93×10^{-4}	2.55×10^{-3}	1.68×10^{-3}	6.04×10^{-3}

Compensated spectra





$$\eta_{\perp\Phi_1} = 2\nu \int_0^\infty k_\perp^4 E_{\Phi_1}(k_\perp)\ dk_\perp$$

$$\varepsilon_{\perp\Phi_1} = 2\nu \int_0^\infty k_\perp^2 E_{\Phi_1}(k_\perp)\ dk_\perp$$

Conjecture of $\Phi_1(k_\perp)$ spectra

$$E_{\Phi_1}(k_\perp) = \begin{cases} \alpha \ \ \eta_{\perp\Phi_1}^{2/3} \ \ k_\perp^{-3} & (k_\perp < k_c) \\ C_K \ \ \varepsilon_{\perp\Phi_1}^{2/3} k_\perp^{-5/3} & (k_\perp > k_c) \end{cases}$$

where

$$\eta_{\perp\Phi_1} = 2\nu \int_0^\infty k_\perp^4 E_{\Phi_1}(k_\perp) \ dk_\perp$$

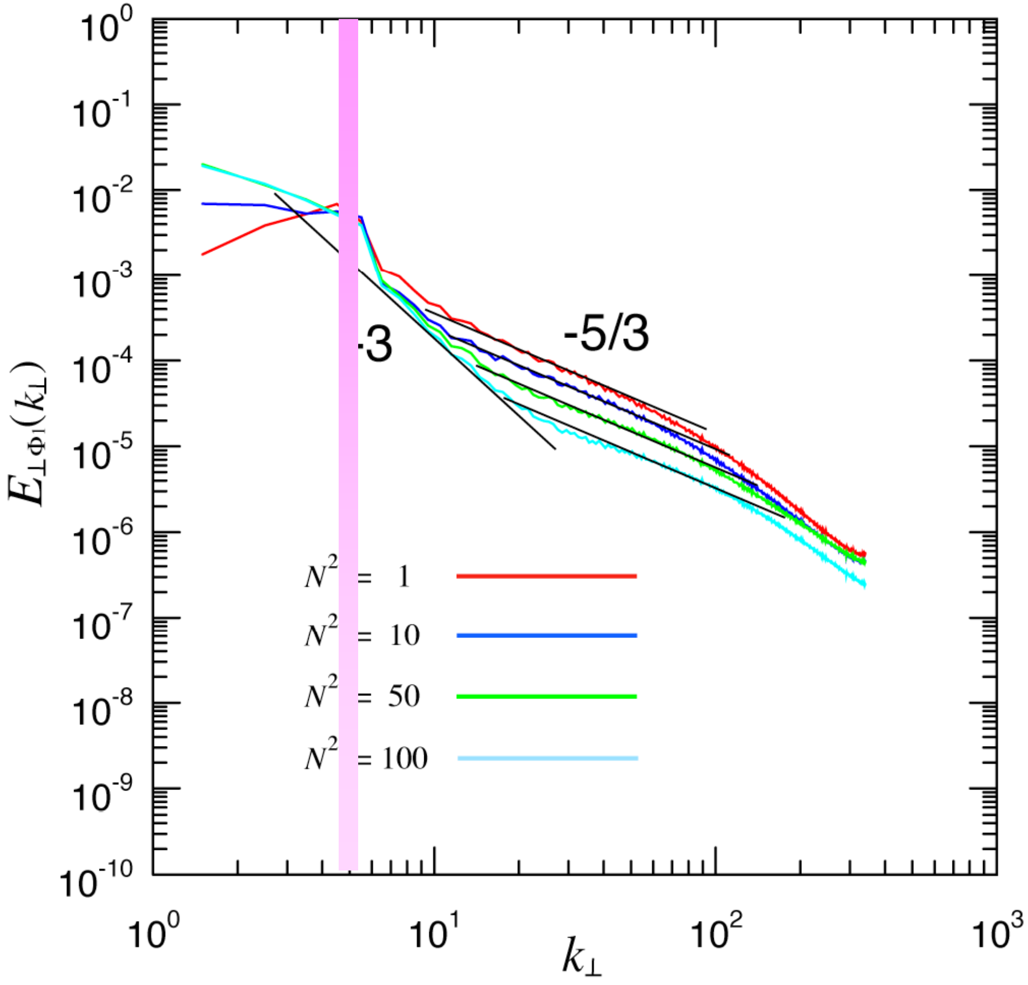
(horizontal enstrophy dissipation)

$$\varepsilon_{\perp\Phi_1} = 2\nu \int_0^\infty k_\perp^2 E_{\Phi_1}(k_\perp) \ dk_\perp$$

(horizontal energy dissipation)

Transition wavenumber

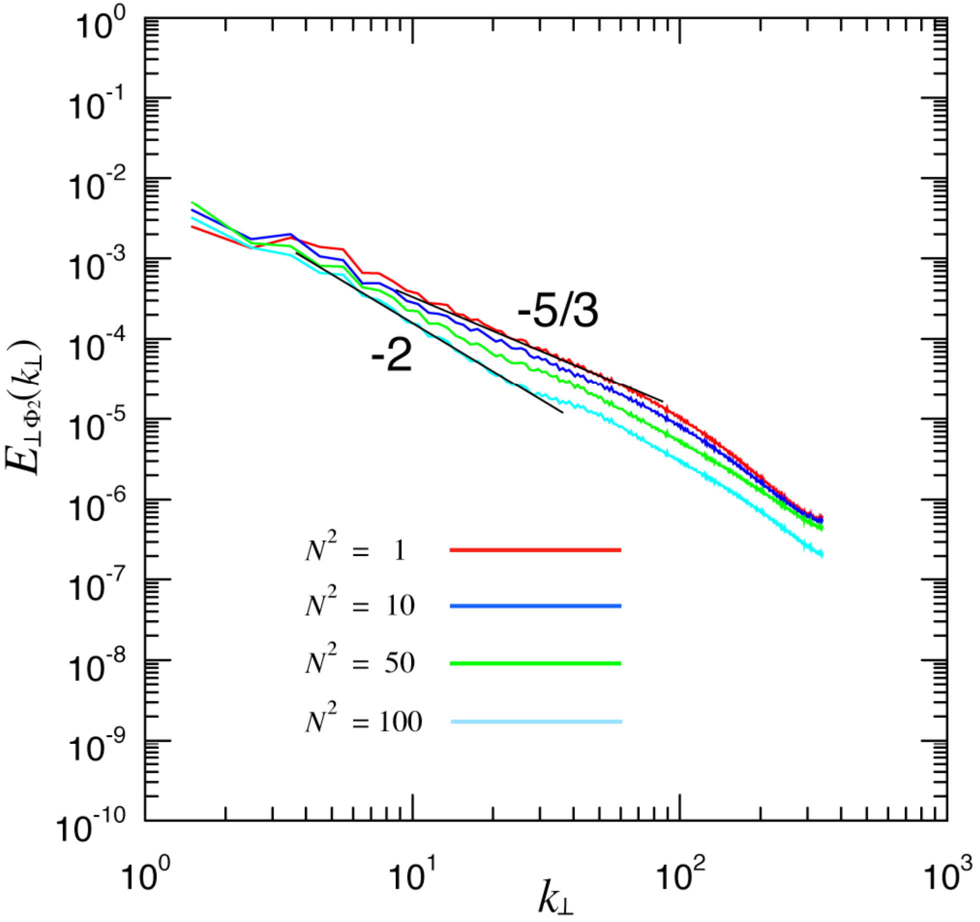
$$\alpha \ \eta_{\perp\Phi_1}^{2/3} \ k_c^{-3} = C_K \ \varepsilon_{\perp\Phi_1}^{2/3} k_c^{-5/3} \quad \Rightarrow \quad k_c = \left(\frac{\alpha}{C_k}\right)^{3/4} \sqrt{\frac{\eta_{\perp\Phi_1}}{\varepsilon_{\perp\Phi_1}}}$$



N^2	k_c
100	11.89
50	10.66
10	8.54
1	7.56

$\Phi_2(k_\perp)$ spcetra for various N

$\Phi_2(k_\perp)$



$$E_{\Phi_2}(k_\perp) = \begin{cases} \beta \sqrt{N\varepsilon_{\perp\Phi_2}}\; k_\perp^{-2} & (k_\perp < k_c) \\ C_K\; \varepsilon_{\perp\Phi_2}^{2/3} k_\perp^{-5/3} & (k_\perp > k_c) \end{cases}$$

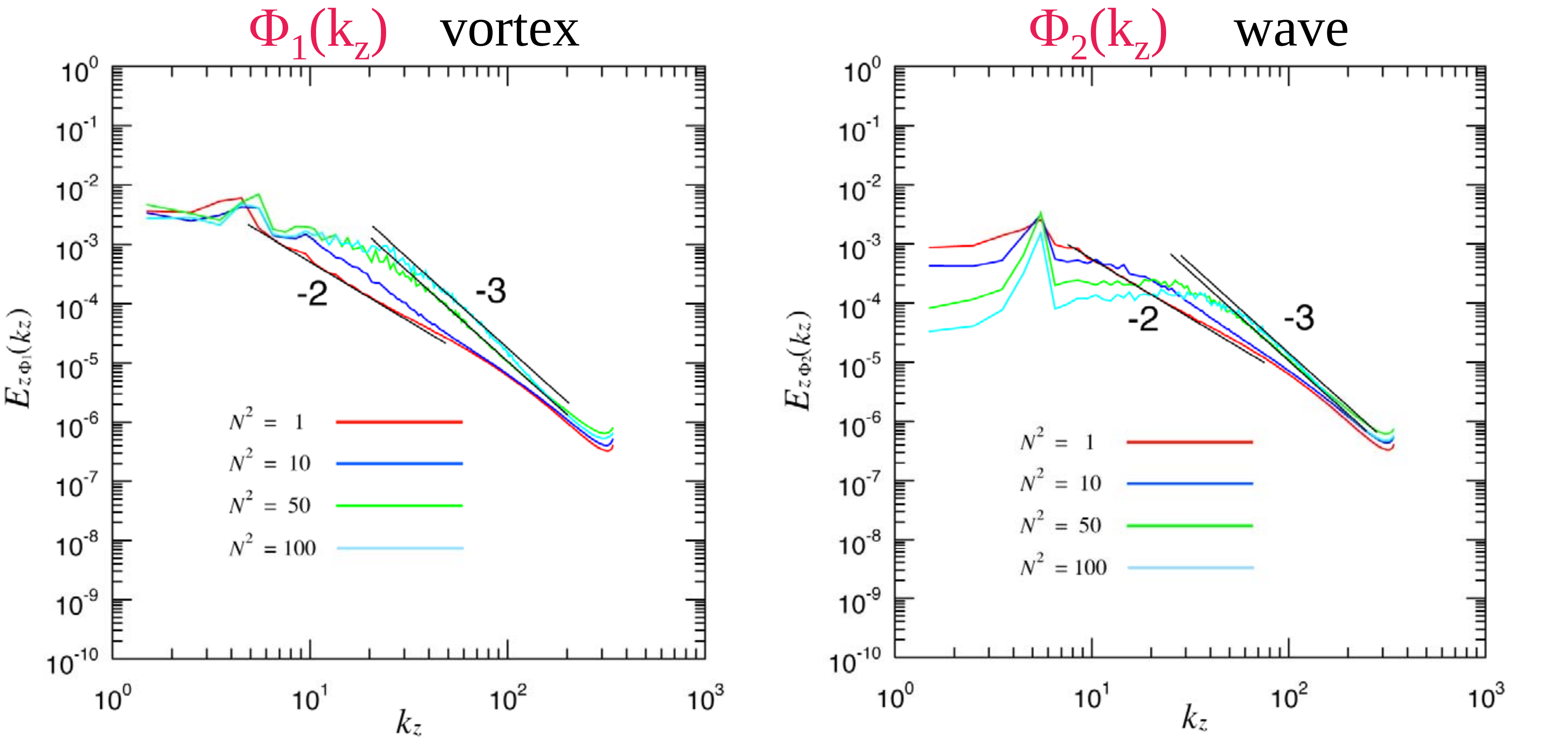
$$k_c = \left(\frac{\beta}{C_k}\right)^3 \sqrt{\frac{N^3}{\varepsilon_{\perp\phi_2}}}$$

transition wavenumber

Proportional to the Ozmidov scale (based on horizontal dissipation)

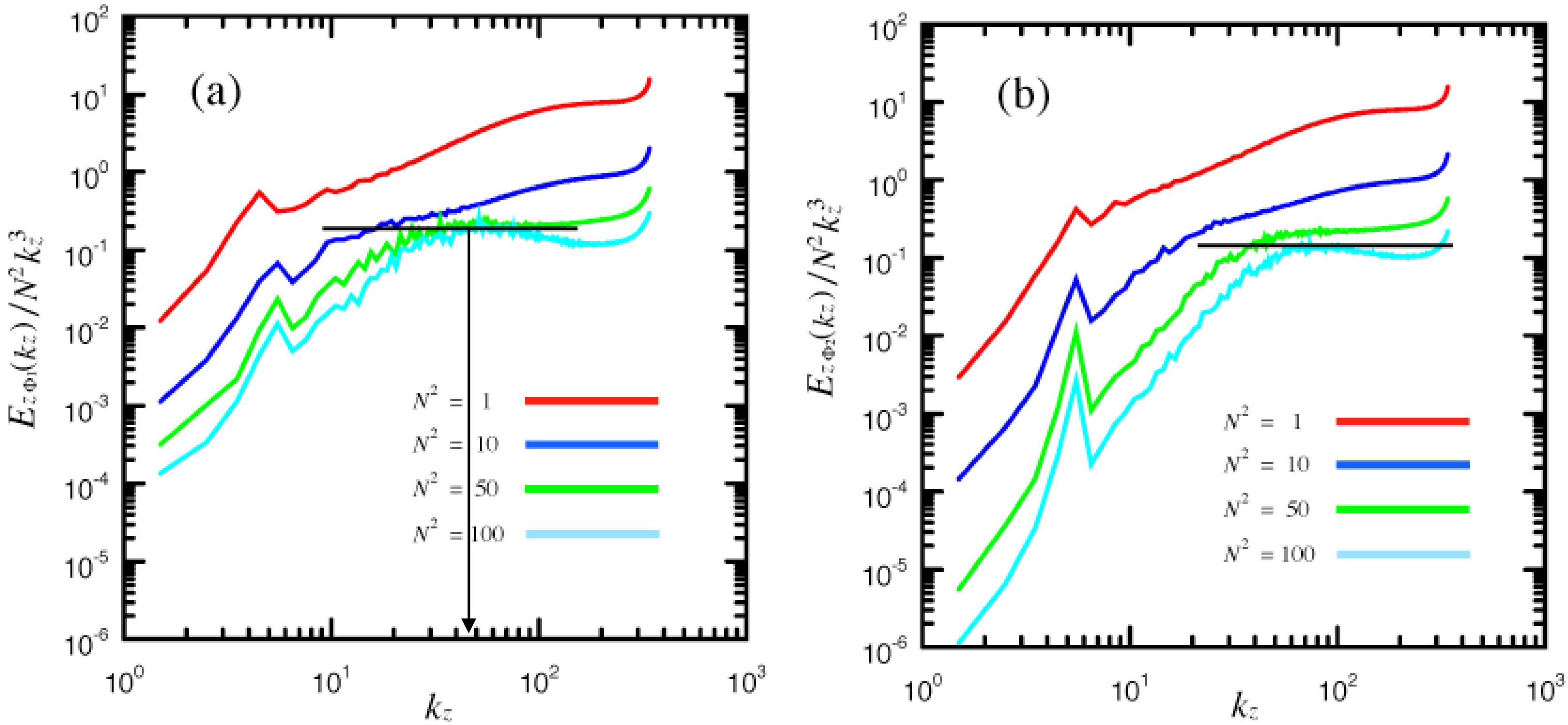
N^2	k_c
100	6.02
50	2.63
10	0.35
1	3.31×10^{-2}

$\Phi_1(k_z)$ & $\Phi_2(k_z)$ spcetra for various N



- flat spectra at large scales
- steep (-3) spectra at small scales for strong stratification

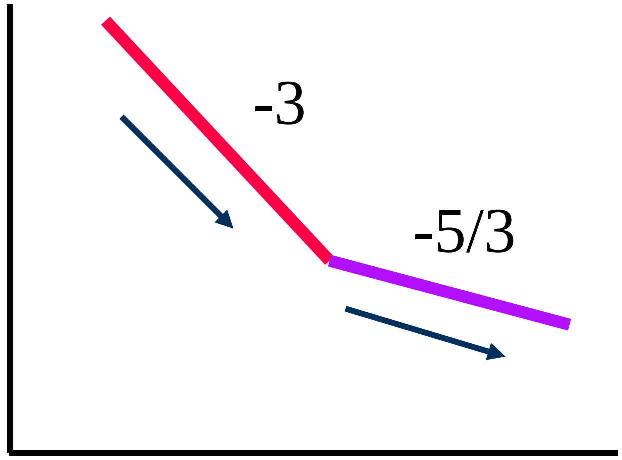
Compensated spectra for $\Phi_1(k_z)$ & $\Phi_2(k_z)$



- starting wavenumber for -3: ~ N/u_{rms}
- coefficient : ~ 0.1 -> $0.1N^2k_z^{-3}$
  (saturation spectra ?)

N^2	N/u_{rms}
100	48.47
50	33.80
10	17.63
1	5.74

More than 1 inertial range



$$\frac{\partial}{\partial t} E_K(k_\perp) = T_K(k_\perp) + B(k_\perp) + D_K(k_\perp) + F_K(k_\perp)$$

$$\begin{cases} T_K(k_\perp) = -\mathrm{Im} \sum\limits_{\mathbf{k}+\mathbf{p}+\mathbf{q}=\mathbf{0}} P_{ijm}(\mathbf{k})\ \tilde{u}_i(\mathbf{k})\tilde{u}_j(\mathbf{p})\tilde{u}_m(\mathbf{q}) \\ B(k_\perp) = \mathrm{Re} \sum\limits_{\mathbf{k}+\mathbf{p}=0} P_{i3}(\mathbf{k})\tilde{u}_i(\mathbf{k})\tilde{\theta}(\mathbf{p}) \\ D_K(k_\perp) = -\nu k^2 E_K(k_\perp) \end{cases}$$

Kolmogrov乱流 → $$\int_0^{k_\perp} T_K(k_\perp)\ dk_\perp \equiv -\ \phi \qquad \text{(flux)}$$

地球流体乱流 → $$\int_{a_i}^{b_i} T_K(k_\perp)\ dk_\perp \equiv -\ \phi_i \qquad (i^{th}\ \text{flux})$$

?

Summary

- Energy spectra are investigated for stably stratified turbulence with 1024^3 pseudospectal DNS simulations.

- Horizontal spectra show clear transition from 2D to 3D Kolmogorov spectra.

- Horizontal spectra are scaled by anisotropic dissipation of energy and enstrophy.

- Vertical spectra show a flat part at large scales and tend to have steeper spectrum(-3) as N becomes large.